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FUNCTIONAL NORMALIZATION FOR THE SOLUTION OF BOUNDARY-VALUE PROBLEMS FOR CYLINDRICAL SHELLS†

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A simple and efficient method of computing the stress–strain state of layered orthotropic cylindrical shells based on the known analytic solution is proposed. Functional normalization of the fundamental system of solutions of ordinary differential equations removes the difficulties due to the presence of rapidly growing and rapidly decaying solutions and makes it possible to compute shells of arbitrary length and thickness.

Using the equilibrium equations, the deformation relations [1], and the elasticity relations for a layered orthotropic material [2], we obtain the following system of equilibrium equations in terms of displacements

$$JU = R^2 h^{-1} \mathbf{q} \tag{1}$$

where $\mathbf{U} = (u, \upsilon, w)^T$ is the displacement vector, $\mathbf{q} = (q_1, q_2, -q_3)^T$ is the external load vector, the matrix components of J have the form

$$J_{11} = b_{11}\frac{\partial^2}{\partial\alpha^2} + b_{33}\frac{\partial^2}{\partial\varphi^2}$$

$$J_{12} = J_{21} = (b_{12} + b_{33})\frac{\partial^2}{\partial\alpha\partial\varphi}, \quad J_{13} = J_{31} = b_{12}\frac{\partial}{\partial\alpha}$$

$$J_{22} = b_{22}\frac{\partial^2}{\partial\varphi^2} + b_{33}\frac{\partial^2}{\partial\alpha^2} + a^2\left(d_{22}\frac{\partial^2}{\partial\varphi^2} + 4d_{33}\frac{\partial^2}{\partial\alpha^2}\right)$$

$$J_{23} = J_{32} = b_{22}\frac{\partial}{\partial\varphi} - a^2\left((d_{12} + 4d_{33})\frac{\partial^3}{\partial\alpha^2\partial\varphi} + d_{22}\frac{\partial^3}{\partial\varphi^3}\right)$$

$$J_{33} = b_{22} + a^2\left(d_{11}\frac{\partial^4}{\partial\alpha^4} + (2d_{12} + 4d_{33})\frac{\partial^4}{\partial\alpha^2\partial\varphi^2} + d_{22}\frac{\partial^4}{\partial\varphi^4}\right)$$

$$a^2 = \frac{h^2}{12R^2}, \quad b_{ij} = \frac{B_{ij}}{h}, \quad d_{ij} = D_{ij}\frac{12}{h^3}$$

and $\alpha = s/R$ and φ are the dimensionless longitudinal and angular coordinates.

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On separating the variables with the aid of a Fourier expansion with respect to the angular coordinate [1], we obtain a system of ordinary differential equations. We will seek a solution of the homogeneous part of this system in the form

$$u_k = A_k e^{\lambda\alpha}, \quad \upsilon_k = B_k e^{\lambda\alpha}, \quad w_k = C_k e^{\lambda\alpha} \tag{2}$$

After this substitution we obtain a system of homogeneous linear algebraic equations for $A_{k'}$, $B_{k'}$, C_k for each kth term of the Fourier series. The condition for non-trivial solutions of the system to exist (its determinant being equal to zero) yields the characteristic equation

$$\begin{aligned}
&\lambda^8 + a_*\lambda^6 + b_*\lambda^4 + c_*\lambda^2 + d_* = 0\\
&a_* = -k^2(F_1 - 2f_{12} + 2g_{12} + 4g_{33})\\
&b_* = k^4[2(F_1 - 2f_{12})(g_{12} + 2g_{33}) + f_{22} + g_{22}] + 2k^2(g_{12} + 4g_{33})(f_{12} - F_1) + F_2 / a^2\\
&c_* = -k^2(k^2-1)^2(4f_{22}g_{33} + F_1 g_{22}) - 2k^4(k^2-1)(f_{22}g_{12} - f_{12}g_{22})\\
&d_* = k^4(k^2-1)^2 f_{22}g_{22}\\
&f_{12} = \frac{b_{12}}{b_{11}}, \quad f_{22} = \frac{b_{22}}{b_{11}}, \quad f_{33} = \frac{b_{33}}{b_{11}}, \quad g_{12} = \frac{d_{12}}{d_{11}}, \quad g_{22} = \frac{d_{22}}{d_{11}}, \quad g_{33} = \frac{d_{33}}{d_{11}}\\
&F_1 = \frac{b_{11}b_{22} - b_{12}^2}{b_{11}b_{33}}, \quad F_2 = \frac{b_{11}b_{22} - b_{12}^2}{b_{11}d_{11}}
\end{aligned} \tag{3}$$

This equation involves only even powers of the characteristic exponents λ to be determined. It can therefore be reduced to an equation of the fourth degree, which can be solved by Ferrari's formulae. For each root λ_i $(i = 1, \ldots, 8)$ the numbers $A_k^{(i)}$, $B_k^{(i)}$, $C_k^{(i)}$ can be determined apart from a multiplier, and the general solution of the homogeneous system of equations can be obtained in the form

$$\begin{Vmatrix} u_k(\alpha) \\ \upsilon_k(\alpha) \\ w_k(\alpha) \end{Vmatrix} = \sum_{i=1}^{8} C_i \begin{Vmatrix} A_k^{(i)} \\ B_k^{(i)} \\ C_k^{(i)} \end{Vmatrix} \exp(\lambda_i \alpha) \tag{4}$$

where C_i are arbitrary integration constants. The cases $k = 0$ and $k = 1$ when four of the eight roots of (3) are zeros, are the only exceptions. The part of the general solution of the homogeneous system of equations corresponding to these roots is:

for $k = 0$

$$\begin{Vmatrix} u_0(\alpha) \\ \upsilon_0(\alpha) \\ w_0(\alpha) \end{Vmatrix} = C_1 \begin{Vmatrix} \alpha \\ 0 \\ -\dfrac{f_{12}}{f_{22}} \end{Vmatrix} + C_2 \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix} + C_3 \begin{Vmatrix} 0 \\ 1 \\ 0 \end{Vmatrix} + C_4 \begin{Vmatrix} 0 \\ \alpha \\ 0 \end{Vmatrix}$$

for $k = 1$

$$\begin{Vmatrix} u_1(\alpha) \\ \upsilon_1(\alpha) \\ w_1(\alpha) \end{Vmatrix} = C_1 \begin{Vmatrix} -2\alpha \\ -\alpha^2 \\ 2\dfrac{f_{12}}{f_{22}}+\alpha^2 \end{Vmatrix} + C_2 \begin{Vmatrix} 6F_1 f_{22} - 3\alpha^2 \\ -\alpha^3 \\ 6\dfrac{f_{12}}{f_{22}}\alpha+\alpha^3 \end{Vmatrix} + C_3 \begin{Vmatrix} 0 \\ -1 \\ 1 \end{Vmatrix} + C_4 \begin{Vmatrix} -1 \\ -\alpha \\ \alpha \end{Vmatrix}$$

Once the displacements are determined, the forces can be computed using the deformation and elasticity relations.

Calculations were carried out for loads given by polynomials of a certain degree along the cylinder generators. In this case it is convenient to seek a particular solution of the inhomogeneous system of ordinary differential equations for the kth term of the Fourier series by the method of undetermined coefficients.

The numerical experiment revealed some characteristic features of the analytic solution.

Firstly, for certain values of the generalized stiffness b_{ij} and d_{ij} the roots of Eq. (3) are not only complex numbers (as in the case of an isotropic body), but also pure real and/or pure imaginary. For example, for the sixth and higher terms of the Fourier series with $R/h = 100$, $d_{11} = b_{11}$, $f_{12} = g_{12} = 0.02$, $f_{22} = g_{22} = 0.6$, $f_{33} = g_{33} = 0.04$ (i.e. in the case when the displacement stiffness of the material is small compared to the expansion stiffness) each of the eight roots of the characteristic equation is a real number.

Secondly, if the shells under consideration are long, it turns out, as a result of the direct application of (4), that the matrix of the system of linear algebraic equations used to determine the arbitrary constants from the boundary conditions is ill-posed due to the presence of rapidly growing and rapidly decaying functions among the solutions. Existing methods of solving this problem (the pivotal-condensation method with orthogonalization and normalization, etc.) require very long computations, leading to long computing times and inefficient use of computer memory.

The above-mentioned difficulty is obviously related only to the features of computer use. But, as opposed to this, in those cases when the volume of computations admits of computation "by hand" the presence of rapidly varying analytic functions simplifies the task because the boundary effect turns out to be the stress state characteristic for shells, and the influence of the conditions on the shell faces on one another can be neglected. It turned out to be very easy to use these ideas in computer calculations. We know [1] that only the decaying solutions (which correspond to roots of the form $\lambda_1 = -p + iq$, $p > 0$) play a role on the left-hand face of a long shell, while only the increasing ones (which correspond to roots of the form $\lambda_2 = p + iq$, $p > 0$) play a role on the right-hand face. From the mathematical point of view this means that if the coordinates of the left and right faces of the shell are $\alpha_{\min}$ and $\alpha_{\max}$, the order of magnitude of any constant corresponding to the root λ_2 is $\exp[p(\alpha_{\max} - \alpha_{\min})]$ times smaller than that of any constant corresponding to λ_i. This obviously furnishes the key to solving the problem: $\exp(\lambda_i\alpha)$ must be replaced by $\exp[\lambda_i(\alpha - \alpha_{\min})]$ in rapidly decaying solutions and by $\exp[\lambda_i(\alpha - \alpha_{\max})]$ in rapidly growing ones.

This substitution (in essence the functional normalization of the fundamental system of solutions) corresponds to choosing new arbitrary constants. All of them turn out to be approximately of order one, the matrix of the system of linear algebraic equations being well posed for an arbitrarily long shell and any kth term of the Fourier series.

In this way we have found an efficient method of computing orthotropic layered cylindrical shells of arbitrary length and thickness.

As an illustration of the method proposed we present the results of solving the problem of the stress-strain state of a layered cylindrical shell supported as a cantilever ($R/h = 100$, $l/R = 1$) with the free end loaded by a radial concentrated force directed towards the axis of the shell. Structures with eight layers have been considered, in each of which the reinforcing fibres are placed at a certain angle β_i, $i = 1, \ldots, 8$ (Tornell-300 material, $E_1 = 142.8\times10^9$ Pa, $E_2 = 9.13\times10^9$ Pa, $G_{12} = \times10^9$ Pa, $\nu_{12} = 0.02$, $\nu_{21} = 0.32$[2]). Each layer has the same thickness, the laminate being symmetric about the middle surface. The generalized

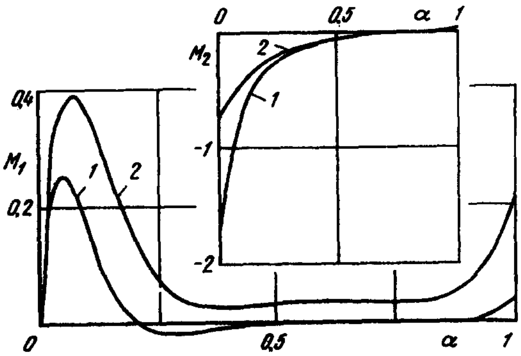

Fig. 1.

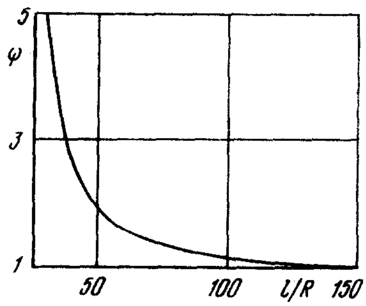

Fig. 2.

stiffnesses are computed from known formulae [2].

Figure 1 shows the results of computing the distribution of the moments M_1 and M_2 along the cylinder generatrix ($\varphi = 0$) for two different reinforcements

$$(1)\ \beta_1 = \beta_8 = 90°,\ \beta_2 = \beta_7 = 45°,\ \beta_3 = \beta_6 = 22.5°,\ \beta_4 = \beta_5 = 0°;$$
$$(2)\ \beta_1 = \beta_8 = 0°,\ \beta_2 = \beta_7 = 22.5°,\ \beta_3 = \beta_6 = 45°,\ \beta_4 = \beta_5 = 90°.$$

In the case in question M_2 has a singularity in the zone in which the concentrated force is applied. Therefore, when determining the inner force factors it proves more convenient to consider the load distributed over an arc of measure, say, 1°, instead of the force. The results are presented in dimensionless form (relative to the product of the load intensity and the shell thickness). When the structure changes one can observe a substantial change in the distribution of the inner force factors: in the latter case, in which the stiffness of the laminate along the circumference decreases, the moment M_2 also decreases, while M_1 increases.

In the case of long cylinders the proposed method enables one to compare the results obtained from the theory of shells and rods. Such an analysis of the reliability of the computational results is practically impossible when other methods are used. In Fig. 2 we show a graph of the ratio ψ of the vertical displacement (at the point of application of the force) of an isotropic shell and $Pl^3/(3EJ)$, which is the maximum deflection of a tubular beam supported as a cantilever for

$$\frac{R}{h} = 300, \quad J = \frac{\pi}{64}[(2R+h)^4 - (2R-h)^4]$$

For $l > 125R$ the maximum displacements computed from these two theories differ by less than 5%.

It follows that the proposed method of functional normalization of the fundamental system of solutions of differential equations considerably simplifies the problem of studying the stress–strain state of a cylindrical shell for arbitrary values of the geometric parameters. Unlike other methods, the resulting form of the analytic solution enables one, even for long and thin shells, to reduce the problem to solving systems of equations of the eighth order, which corresponds to the number of boundary conditions, without any additional techniques. Moreover, computer calculations can be performed with guaranteed accuracy.

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